VII. *Forced Oscillations in a Circuit with non-linear Resistance. (Reception with reactive Triode.)* *By* BALTH. VAN DER POL, Jun., *D.Sc.*[*]

§ 1. *Introduction.*

WHEN an E.M.F. $B \sin \omega t$ acts on a circuit consisting of a self-inductance L, capacity C, and ohmic resistance r, free and forced oscillations are generally set up, the amplitude b of the forced oscillation being given by

$$b = \frac{B}{\sqrt{\left(L\omega - \frac{1}{C\omega}\right)^2 + r^2}}.$$

Should the system, in addition, be in resonance with the external E.M.F. so that

$$L\omega - \frac{1}{C\omega} = 0,$$

the resultant amplitude would have the value

$$b = \frac{B}{r}.$$

It follows from this that when r approaches zero value, the amplitude b would become infinite.

A circuit electrically coupled to a triode with reaction may be regarded as a system the resistance of which varies with the reaction, the resistance being positive when the reaction coupling is loose. The circuit will commence to oscillate, however, when the point of critical reaction is passed and would then act as if the resistance were negative. At the critical value of the reaction the resistance is zero. (The term resistance here denotes what might conveniently be called the "differential resistance" as defined by $\frac{dV}{di}$, which expression has the dimension of a resistance analogous to $\frac{V}{i}$).

According to the above elementary consideration, should the critical point of the reaction be approached, even the least external E.M.F., tuned to the frequency of the circuit, would set up forced oscillations, the amplitude of which would tend to increase to infinity. This is of course contrary to actual experience.

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The bends of the characteristics of the triode limit in practice the resultant amplitude. We are therefore forced to consider the resistance of the circuit as being dependent on the existing amplitude.

The deduction of this non-linear resistance from the triode characteristics has already been fully shown in previous publications * and need not be further gone into at present. If the coupling is increased past the critical reaction coupling, the resistance becomes negative and, considered linearly, the amplitude of the forced oscillation should commence to decrease again in accordance with the equation:

$$b=\frac{\mathrm{B}}{\sqrt{(-r)^2}}=\frac{\mathrm{B}}{r}.$$

Linearly considered, therefore, a negative resistance would not impart greater signal strength than a positive resistance of equal magnitude, and in order to obtain maximum signal strength it would apparently only be necessary to reduce the resistance to a minimum.

The free oscillations, however, which automatically fade away in the case of a positive resistance, increase exponentially when the resistance is negative. The interference of the free oscillations would become apparent for the first time at the moment the resistance becomes zero.

The experiments by Vincent, Möller, Miss Leishon, Mercier, Rossman and Zenneck †, which are fully confirmed by accurate observations made on wireless reception using a reactive triode, show, on the contrary, that a circuit *does not* oscillate spontaneously when the reaction passes the critical point of a triode system, when the latter is tuned for the reception of signals.

It can be demonstrated especially in the case of wireless telephonic reception of a carrier wave, that the reaction may be brought past the critical point without spontaneous free oscillations being set up in the circuit, which is under the influence of the carrier wave. As soon as the influence of the carrier wave ceases, the circuit commences to generate free oscillations. This phenomenon may also be observed in the following simple way.

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* Van der Pol, *Tijdschr. v. h. Ned. Radiogen.* i. p. 1 (1920). Van der Pol, Radio Review, i. p. 701 (1920). Appleton and Van der Pol, Phil. Mag. xlii. p. 201 (1921).

† Vincent, Proc. Phys. Soc. Lond. xxxii. p. 84 (1920). Möller, *Jahrb. f. drahtl. Tel.* xvii. p. 269 (1921). Miss Leishon, Phil. Mag. xlvi. p. 686 (1923). Mercier, *C. R.* clxxiv. p. 448 (1922). Zenneck, *Jahrb. f. drahtl. Tel.* xxiii. p. 47 (1924).

A receiving circuit will oscillate spontaneously when detuned during the reception of a carrier wave, or during telephonic reception, when the reaction coupling is greater than the critical. This is proved by the production of an audible combination tone built up by the detection (quadratic terms) from the free and forced oscillations.

If the tuning is now brought closer to resonance, the combination tone will become lower and then suddenly disappear, though the frequency still differs from zero. The combination tone is therefore not heard close to the point of resonance. This is not due to the frequency being too low to be heard, but to the actual disappearance of this beat note, caused by the *absence of free oscillations,* forced oscillations only being present in the circuit.

The explanation of this essentially non-linear phenomenon must be sought in the fact that the representative point passes at a given period through a section of the triode-characteristic where it encounters a smaller slope, so that the resistance is no longer negative but positive. This occurs with forced oscillations when the reaction coupling is greater than the critical. The effective influence is that the free oscillation is discouraged to build up, for, what might conveniently be termed "the remaining average resistance," may under these circumstances no longer be negative.

This *suppression of the free by the forced oscillations* must be ascribed to the interaction of both types of oscillation *. This interaction is entirely absent in linear circuits but is, as has already been explained elsewhere, of vital importance in the production of triode oscillations. There can be no question here of the free oscillations being "taken along by" the forced oscillation, as is sometimes expressed in articles on this subject, although in passing we may as well point out that the presence of forced oscillations renders the free oscillations subject to a frequency-correction, which behaves as if the forced "attracted" the free frequency.

Six years ago the problem of forced oscillations in a circuit with non-linear resistance was investigated for the first time by the author †. The differential equation

$$\ddot{v}+\phi(v)\dot{v}+\omega_0^2 v=\omega_1^2 \mathrm{B} \sin \omega_1 t,$$

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* The synchronous time-keeping of two clocks hung on the same wall, as long ago observed by Huygens, is another example of the suppression of free by forced oscillations.

† Van der Pol, *Tijdschr. v. h. Ned. Radiogen.* i. p. 1 (1920).

deduced at the time, again forms the basis of the present investigation. Our remarks, however, were then confined solely to dealing with the case where the resistance remained positive. At a later date and in collaboration with Dr. Appleton, the general theoretical problem where the resistance could also be negative was investigated, the results of which are to be found in an article by Dr. Appleton *. The present article, however, is more general and gives a more detailed account of the experiments.

In conclusion it should be pointed out that, although the conception of free and forced oscillations is essentially linear, this interpretation has been adhered to in the present non-linear investigation in view of the fact that, when the logarithmic increments or decrements are small, as is generally the case, the resistance-correction of the frequency would be negligible and the non-linear resistance characteristic would merely find expression as a first approximation in limiting the amplitudes.

§ 2. The differential equation for the anode voltage of a triode receiver acted upon by an external periodic E.M.F. is expressed by

$$\ddot{v}+\phi(v)\dot{v}+\omega_0^2 v = \omega_1^2 \mathrm{B}\sin\omega_1 t, \quad . \quad . \quad . \quad (1)$$

in which B denotes the amplitude of the external E.M.F., due for example to a signal, $\omega_0^2=\frac{1}{\mathrm{CL}}$ the square of the angular frequency of the receiver, and ω_1 the angular frequency of the external E.M.F.

Furthermore,

$$\int\phi(v)dv = \frac{rv}{\mathrm{L}}+\frac{\psi(v)}{\mathrm{C}},$$

in which r denotes the ohmic resistance of the L, C, r circuit, and $\psi(v)$ the "oscillation characteristic" † of the reactive triode.

Eliminating all secondary considerations having no direct bearing on the present investigation, the problem resolves itself into the study of an electric disturbance in a circuit as indicated in fig. 1.

To simplify matters it is assumed that the triode is adjusted to work on a symmetrical portion of the oscillation characteristic, such that no even terms ‡ occur in the powers

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* Appleton, Proc. Cambr. Phil. Soc. xxiii. p. 231 (1923).

† Appleton and Van der Pol, Phil. Mag. xliii. p. 179 (1921).

‡ The quadratic term βv^2 is of no importance in the first approximation, when considering the amplitudes. The principal influence of the term βv^2 is found in detection and modulation.

of the series for $\psi(v)$, and we confine ourselves to the terms:

$$\phi(v) = -\alpha + 3\gamma v^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

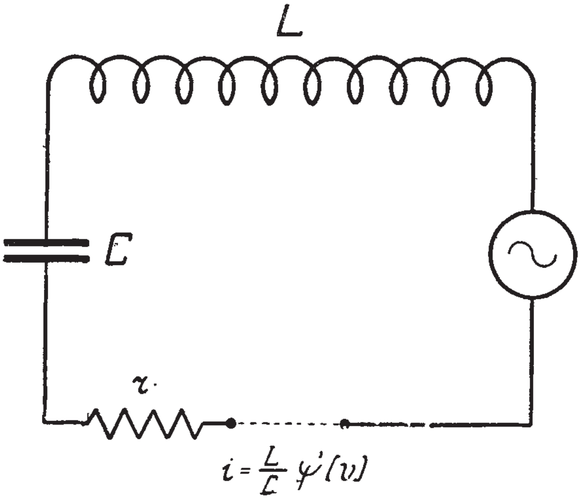
in which α and γ are positive quantities.

The fundamental differential equation now becomes

$$\ddot{v} - \alpha\dot{v} + \gamma\frac{dv^3}{dt} + \omega_0^2 v = B\omega_1^2 \sin\omega_1 t. \quad . \quad . \quad . \quad (1)$$

If the third non-linear term, $\gamma\frac{dv^3}{dt}$, were absent, formula (1) is reduced to the well-known differential equation for forced oscillations.

Fig. 1.



The new term γ, which follows logically from the characteristic curves, suddenly introduces all the peculiarities discussed in § 1.

§ 3. The case wherein the receiving circuit is tuned, or nearly so, to the received oscillations is of prime interest. Hence we try for the general solution of (1) the expression

$$v = b_1 \sin\omega_1 t + b_2 \cos\omega_1 t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which b_1 and b_2 are functions of t but only slowly variable, viz.:

$$\dot{b}_1 << \omega_1 b_1,$$
$$\dot{b}_2 << \omega_1 b_2,$$

and

$$\ddot{b}_1 << \omega_1 \dot{b}_1,$$
$$\ddot{b}_2 << \omega_1 \dot{b}_2,$$

so that $\ddot{b}_1$ and $\ddot{b}_2$ may be neglected.

The solution allows for the possibility of the presence of free oscillations, by the assumption that both b's are functions of t. This will be explained more fully further on. In the term v^3 we only retain the fundamental frequency such that

$$v^3 = \tfrac{3}{4}(b_1^2+b_2^2)(b_1\sin\omega_1 t+b_2\cos\omega_1 t). \quad . \quad . \quad (4)$$

By substituting (3) and (4) in (1) and separately equating the terms containing $\sin\omega_1 t$ and $\cos\omega_1 t$ to zero, we have:

$$\left.\begin{aligned} 2\dot{b}_1+zb_2-\alpha b_1\left(1-\frac{b^2}{a_0^2}\right)&=0,\\ 2\dot{b}_2-zb_1-\alpha b_2\left(1-\frac{b^2}{a_0^2}\right)&=B\omega_1,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which

$$\left.\begin{aligned} z&=\frac{\omega_0^2-\omega_{12}}{\omega_1}\doteq 2(\omega_0-\omega_1),\\ b^2&=b_1^2+b_2^2,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$a_0^2=\frac{\alpha}{\frac{3}{4}\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

represents the amplitude of the free stationary oscillation.

§ 4. Before completing the solution of the non-linear case, two particular cases will be investigated first. In the first case we will investigate whither the present mode of solution, which differs from those usually adopted, will lead us in a linear case. To this end we assume in (7) that

$$\gamma = 0,$$

so that a_0^2 becomes ∞.

This has the effect of simplifying (5) to

$$\left.\begin{aligned} 2b_1+zb_2-\alpha b_1&=0,\\ 2b_2-zb_1-\alpha b_2&=B\omega_1\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

the solutions of which are:

$$\left.\begin{aligned} b_1&=e^{\frac{\alpha}{2}t}\left(C_1\sin\frac{z}{2}t+C_2\cos\frac{z}{2}t\right)-\frac{z\omega_1 B}{z^2+\alpha^2},\\ b_2&=e^{\frac{\alpha}{2}t}\left(-C_1\cos\frac{z}{2}t+C_2\sin\frac{z}{2}t\right)-\frac{\alpha\omega_1 B}{z^2+\alpha^2}\end{aligned}\right\}.$$

b_1 and b_2 are therefore of a slow periodicity. By substituting these in (3) we obtain the following equation for v:

$$v = \mathrm{C}_3 e^{\frac{\alpha}{2}t} \sin(\omega_0 t + \phi) + \frac{\mathrm{B}\omega_1}{\sqrt{(\omega_0 - \omega_1)^2 + \alpha^2}} \sin\left(\omega_1 t + \tan^{-1}\frac{2\alpha}{\omega_0 - \omega_1}\right),$$

which expression agrees with the exact linear solution of (1) if we neglect the frequency correction to which the free oscillations are subjected on account of the resistance. Hence the free oscillations are represented by the slow periodic part of the b's.

Secondly, we will consider the case in which the external E.M.F. B is equated to zero.

Expression (5) then leads to

$$b_1 = \frac{a_0 \cos\left(\frac{z}{2}t + \phi\right)}{\sqrt{1 + \mathrm{C}e^{-\alpha t}}},$$

$$b_2 = \frac{a_0 \sin\left(\frac{z}{2}t + \phi\right)}{\sqrt{1 + \mathrm{C}e^{-\alpha t}}};$$

so that in this case we obtain for the solution of the (only present) free oscillation in (1):

$$v = \frac{a_0 \sin(\omega_0 t + \phi)}{\sqrt{1 + \mathrm{C}e^{-\alpha t}}},$$

which agrees with our previous solution *.

§ 5. Returning again to the solution of the general problem, we find that a particular solution of (5) is given by

$$\dot{b}_1 = \dot{b}_2 = 0,$$

so that

$$\left.\begin{aligned} zb_2 - \alpha b_1\left(1 - \frac{b^2}{a_0^2}\right) &= 0, \\ -zb_1 - \alpha b_2\left(1 - \frac{b^2}{a_0^2}\right) &= \mathrm{B}\omega_1; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

whence

$$z^2 + \alpha^2\left(1 - \frac{b^2}{a_0^2}\right)^2 = \frac{\mathrm{B}^2\omega_1^2}{b^2}. \quad . \quad . \quad . \quad . \quad (9)$$

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* *Tijdschr. v. h. Ned. Radiogen.* i. p. 21 (1920).

In this instance we are dealing solely with the presence of forced oscillation, seeing that both b's in this particular solution are independent of the time. The free oscillations, which are represented by periodic parts of the b's, are here completely absent.

The circumstances under which this condition obtains are determined by a further stability investigation.

To effect this we will consider small variations Δb_1 and Δb_2 from the solutions b_1 and b_2 as obtained from (8), and determine whether or not these deviations approach 0 with the time.

From (5) and (8) follows a linear equation for both Δb's (with $\mathrm{D} = \frac{d}{dt}$):

$$\left[4\mathrm{D}^2 - 4\alpha\mathrm{D}\left(1 - \frac{2b^2}{a_0^2}\right) + \alpha^2\left(1 - \frac{b^2}{a_0^2}\right)\left(1 - \frac{3b^2}{a_0^2}\right) + z^2\right]\left\{\begin{matrix}\Delta b_1 \\ \Delta b_2\end{matrix}\right. = 0. \qquad (10)$$

The solution (10), which has the form

$$x^2 + ex + f = 0,$$

only approaches zero with the time when

$$e > 0 \quad \text{and} \quad f > 0.$$

Hence the solution (8) is only stable when

$$b^2 > \tfrac{1}{2}a_0^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\alpha^2\left(1 - \frac{3b^2}{a_0^2}\right)\left(1 - \frac{b^2}{a_0^2}\right) + z^2 > 0. \quad . \quad . \quad . \quad (12)$$

The first, (11), of these conditions required for the stable solution of (8, 9), *i.e.*, where the free oscillations are suppressed by the forced oscillations, may be put in words as follows:—

> *The development of the free oscillation is suppressed if the square of the resultant amplitude of the forced oscillation is greater than half the square of the amplitude which the free oscillation would attain in the absence of an external E.M.F.*

The second condition, (12), for the suppression of the free by the forced oscillation may also be written

$$\frac{db^2}{d\mathrm{B}^2} > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,a)$$

or

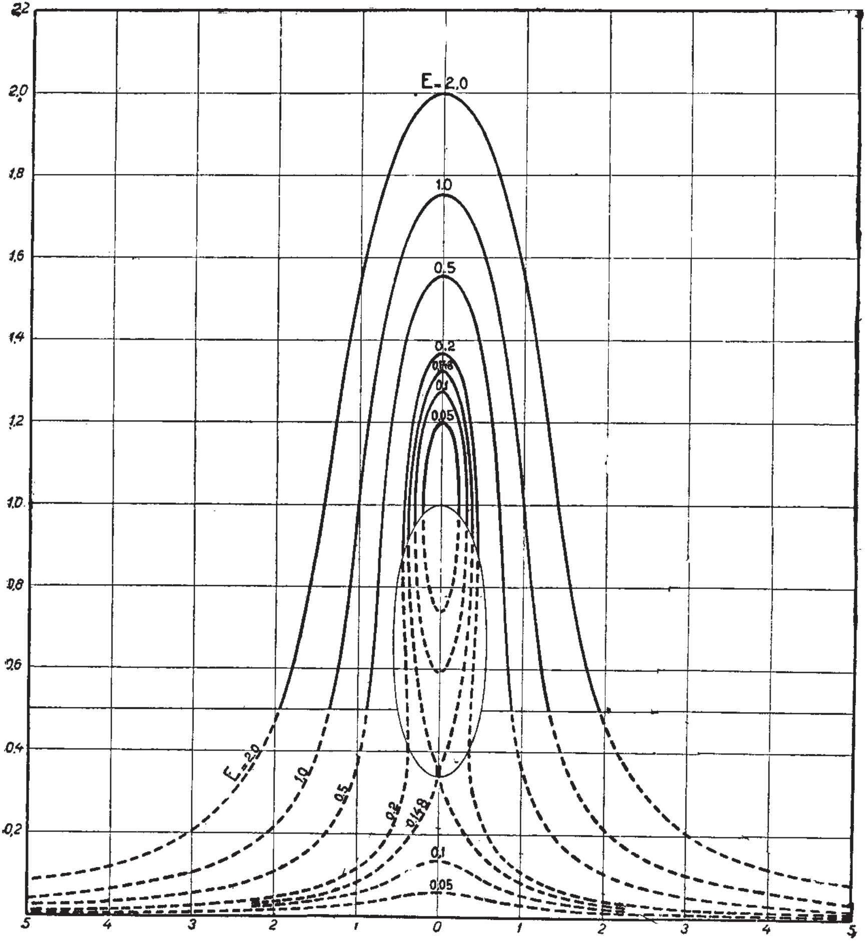
$$\frac{db^2}{dz^2} > 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,b)$$

i. e.:

> *Free oscillations are also suppressed by the forced oscillations if in (9) of the three available values of b^2 of the forced oscillations, such a value is chosen that the amplitude increases with increasing external E.M.F. This is quite a plausible condition.*

Condition (12 *b*) simply lays down that *a stable solution is only obtained provided that an increased amplitude results from an increased accuracy in tuning.*

Fig. 2.



§ 6. The results obtained up to now are illustrated by a few resonance curves given in fig. 2.

To this end (9) may be written

$$x^2+(1-y)^2 = \frac{E}{y}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which

$\frac{z}{\alpha} = \frac{1}{2}x$ represents the amount of detuning,

$\frac{b^2}{a_0^2} = y$ „ the resultant amplitude,

$\frac{\omega_1^2 B}{a_0} = E$ „ the external E.M.F.

The x used in fig. 2 thus measures the degree of detuning, and the ordinate y the resultant amplitude of the forced oscillation for the following values of the external E.M.F.:

E = 0·05 0·1 0·148 0·2 0·5 1·0 and 2·0.

Stable conditions are shown by full lines, whilst unstable conditions are represented by the broken lines. The unstable portions are according to (11) and (12) limited by the curves

$$y = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$(1-3y)(1-y)+x^2 = 0, \quad . \quad . \quad . \quad . \quad (15)$$

which areas are bounded by thin lines in the figure.

Fig. 2 clearly shows that for a strong signal, in which case E=2, the resultant amplitude attains a maximum when the circuit is in tune, dropping off to each side of the maximum in a manner practically similar to the ordinary (linear) resonance curve.

The slope of the resonance curve is, however, smaller than that obtained with a linear resistance for the same resonance amplitude.

The shape of the E=2, E=1, and E=0·5 curves is roughly similar; for weaker signals, *i. e.* when E=0·2 and 0·148, the resonance curves are slightly "waisted." They are, however, only stable up to the point where the slope becomes vertical, as for stability the condition

$$\frac{db^2}{dz^2} > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\,b)$$

must be satisfied.

When the signals are very weak, E=0·1 and 0·05, the resonance curve splits up into two parts, the upper part assuming an approximately elliptical form, the lower part rising to a slight maximum. This lower part, falling below the value $y=\frac{1}{2}$, is unstable as shown by (11), and need not be further discussed ; the upper portion is elliptical and only stable up to the point where the curve becomes vertical.

§ 7. We still have to consider the disturbance occurring outside the limits of stability of $b^2=\text{const.}$ Experiments go to show that the solution is now no longer of single periodicity, and thus may be regarded as the point at which free oscillations set in. Therefore, if an unmodulated carrier wave is being received, nothing will be heard in the region for which a stable solution of $b^2=\text{const.}$ is obtained. As soon, however, as the boundary of the stability is crossed, the familiar heterodyne note will be heard.

Fig. 2 shows that the width of this "silent region" increases with the strength of the incoming signals E. For *weak signals* this width is given by

$$x^2+(1-y)^2 = \frac{E}{y} \quad . \quad . \quad . \quad . \quad (13)$$

and

$$x^2+(1-y)(1-3y) = 0. \quad . \quad . \quad . \quad . \quad (15)$$

If the *signals* are *strong*, the width of the silent region is determined by

$$x^2+(1-y)^2 = \frac{E}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$y = \tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The last expression may also be written

$$(\omega_0-\omega_1)^2+\tfrac{1}{16}\alpha^2 = \frac{\omega_1{}^2 B}{2a_0{}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (m)$$

or, for very strong signals, where

$$\tfrac{1}{16}\alpha^2 < < (\omega_0-\omega_1)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (n)$$

the expression becomes

$$\frac{\omega_0-\omega_1}{\omega_1} = \mp\frac{B}{\sqrt{2}\,.\,a_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Expression (16) is therefore only valid for very strong signals, and has for this case been confirmed experimentally by Dr. Appleton *.

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* Appleton, Proc. Camb. Phil. Soc. *l. c.*

8. The disturbance beyond the limits of the silent region is given by (5), in which the b's are to be taken as functions of the time. We have not so far succeeded in obtaining a general solution for (5). It is possible, however, to determine the point at which free oscillations would set in on the borders of the silent region, by determining the small deviations Δb from b at that point. Thus for weak signals, for which (12) is valid, expression (10) gives us on the borders of the silent region :

$$\Delta b_1 = C_1 + C_2 e^{a\left(1-\frac{2b^2}{a_0{}^2}\right)t}.$$

The second term in this expression diminishes to zero with the time, and the equilibrium at that point is therefore indifferent.

For stronger signals, the stability conditions of which are determined by (11), we obtain in the same way from equation (10) the following expression for Δb_1 at the border :

$$\Delta b_1 = C_3 \sin\left(\frac{1}{2}\sqrt{z^2-\frac{\alpha^2}{4}}\,.\,t+\psi\right)$$

$$= C_3 \sin\left(\sqrt{(\omega_0-\omega_1)^2-\frac{\alpha^2}{16}}\,.\,t+\psi\right).$$

The free frequency with which the circuit commences to oscillate spontaneously at the border is not ω_0, but is expressed by

$$\omega_1-\sqrt{(\omega_0-\omega_1)^2-\frac{\alpha^2}{16}}. \quad . \quad . \quad . \quad . \quad (17)$$

Hence the free frequency undergoes a correction in the direction of the forced frequency, giving the impression as if the free frequency were being attracted by the forced frequency. This correction of the free frequency has experimentally already been noted by Vincent, Möller, and Appleton.

§ 9. As a first approximation of a particular solution of (1) at a point far outside the resonance region, it is permissible to assume a linear combination of the free and forced oscillations, viz. :

$$v = a \sin(\omega_0 t + s) + b \sin(\omega_1 t + \lambda), \quad . \quad . \quad (18)$$

in which a, b, s, and λ are constants.

If only the frequencies ω_0 and ω_1 are retained in v^3, we have

$$v^3 = \tfrac{3}{4}[a(a^2+2b^2)\sin(\omega_0 t+s)+b(b^2+2a^2)\sin(\omega_1 t+\lambda)]. \quad . \quad . \quad . \quad (19)$$

On substituting (18) and (19) in (1), equating the terms containing $\sin\omega_0 t$, $\cos\omega_0 t$, $\sin\omega_1 t$, and $\cos\omega_1 t$ to zero, and neglecting the small terms, we obtain the following four equations :

$$\left.\begin{aligned} \omega_0-\omega_0 &= 0, & &(a)\\ a\left(1-\frac{a^2+2b^2}{a_0^2}\right) &= 0, & &(b)\\ b\,\frac{z}{\omega_1} &= -\mathrm{B}\cos\lambda, & &(c)\\ \frac{\alpha}{\omega}\,b\left(1-\frac{b^2+2a^2}{a_0^2}\right) &= -\mathrm{B}\sin\lambda. & &(d) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

Equation (20 a) shows that the choice of the free oscillation frequency in (18) without correction was right as a first approximation. The symbol s does not appear in (20), which indicates that we are free to make an arbitrary choice in regard to the phase. Expression (20 b) resolves into the following two solutions :

$$a = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$1-\frac{a^2+2b^2}{a_0^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The first solution represents the suppression of the free by the forced oscillation. This is therefore the solution applying to the silent region which has already been fully discussed; for, if (c) and (d) are squared and added together, we obtain equation (9).

Solution (22), on the other hand, admits of a finite amplitude being given to the free oscillation.

By eliminating a^2 from (22), (20 c) and (20 d) we obtain

$$z^2+\alpha^2\left(1-\frac{3b^2}{a_0^2}\right)^2 = \frac{\mathrm{B}^2\omega_1^2}{b^2}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If (23) be compared with (9) it will be seen that owing to the presence of the free oscillations the circuit behaves with regard to the forced oscillations as if γ, which indicates the change in resistance with the amplitude, had been multiplied by 3.

Equation (22) shows moreover that the free amplitude (a^2) cannot develop to $a_0{}^2$ but only to

$$a^2 = a_0{}^2 - 2b^2,$$

from which it follows that, as soon as the forced amplitude equals

$$b^2 = \tfrac{1}{2}a_0{}^2,$$

no room remains for the free oscillation, seeing that a^2 would become negative. As soon as this occurs the free oscillation is suppressed by the forced oscillation. This agrees with the results given by equation (11) indicating the same limiting value.

§ 10. The results obtained in § 9 enable us to extend fig. 2 to the curves given in fig. 3. The mean square of the total disturbance in the presence of free oscillations is given by

$$\tfrac{1}{2}(a^2+b^2).$$

The curves in fig. 3 are, within the silent region, the same as those shown in fig. 2; outside this region, where free oscillations are present, the quadratic amplitude a^2+b^2 has been plotted in accordance with equations (23) and (22).

Fig. 3 shows that a good connexion has been established between solutions (9) and (23), especially where the external E.M.F. is high. However, where

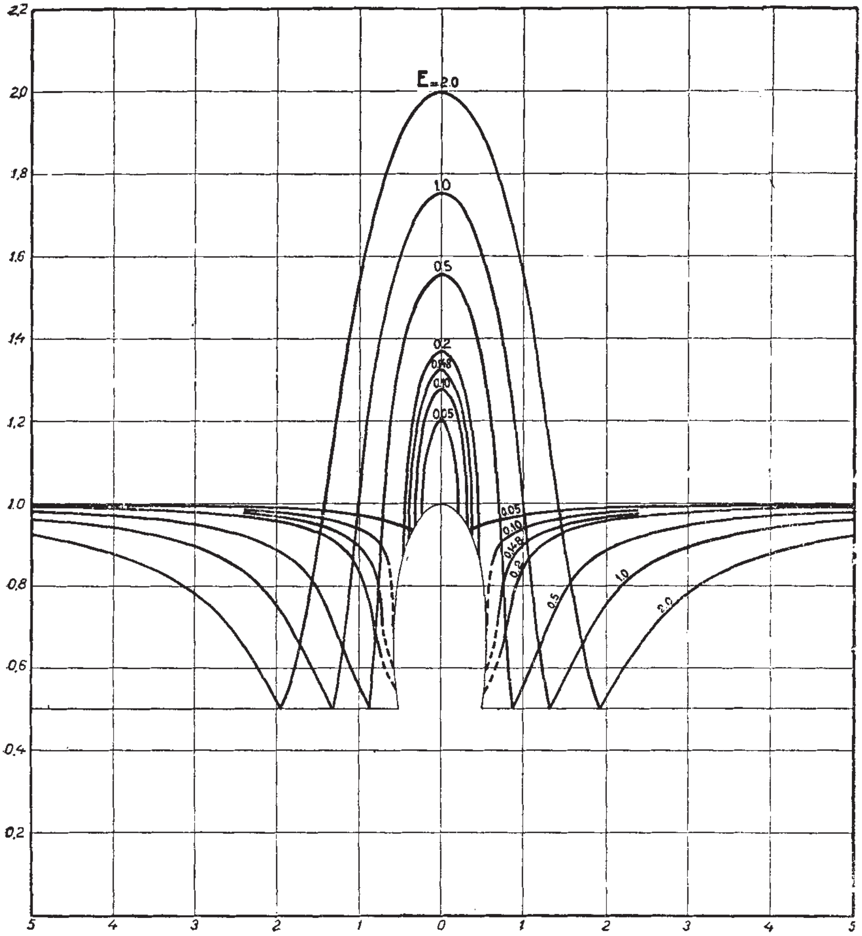
$$E = \frac{\omega_1{}^2 B^2}{a_0{}^2} = 0{\cdot}2 \text{ to } 0{\cdot}05$$

we encounter a small region, shown dotted in the figure, where the connexion is not complete. Only a complete solution of (5) could establish such a connexion. If fig. 3 be further compared with the experimental results obtained by Möller, it will be seen that these are in excellent agreement. (See also J. Golz, *Jahrb. Drahtl. Tel.* xix. p. 281, 1922.)

It may at first appear somewhat strange that the mean square of the disturbance, as shown in fig. 3, is sometimes smaller under the influence of an external E.M.F. than that obtained when free oscillations only are present.

In a physical way, however, this result may be roughly interpreted as follows:—If free oscillations only are present, the amplitude attained is determined by the curvature of the characteristic. If in addition a forced oscillation is present, the representative point will tend to oscillate on such parts of the characteristic where the slope is smaller, resulting in a reduction of the mean negative resistance available for the free oscillations. This effect more than counterbalances the contribution made by the free oscillations to the mean square total amplitude, the net result of which is that the latter decreases in consequence of the simultaneous presence of the two oscillations.

Fig. 3.



Summary.

If a triode system which is reaction-coupled beyond the critical point is acted upon by a signal in the form of a continuous wave, the circuit must satisfy the following equation (secondary factors being eliminated):

$$\ddot{v}-(\alpha-3\gamma v^2)v+\omega_0^2 v = \mathrm{B}\omega_1^2 \sin\omega_1 t, \quad . \quad . \quad . \quad (1)$$

in which the non-linear term containing γ is essential in order to represent the experimental facts. In the absence of an external E.M.F., *i. e.* without the right-hand term, this equation gives us a free oscillation of frequency ω_0 having a constant amplitude a_0 determined by

$$a_0^2 = \frac{a}{\frac{3}{4}\gamma}.$$

If an external E.M.F. is present, the following phenomena are observed:—

Close to the resonance region (ω_1 practically equal to ω_0) forced oscillations only are present the amplitude of which is greater than a_0. At this point the free oscillations are suppressed by the forced oscillations. Moving away from the resonance region, by varying ω_0 the amplitude of the forced oscillation decreases until it falls below a_0. Continuing this detuning, the free oscillations commence to set in with a small amplitude; the latter are subjected, however, to a small frequency correction in the direction of the forced oscillation frequency. The mean square of the total disturbance, consisting of free and forced oscillations, is smaller than that set up by the free oscillations only.

The limits of stability of the suppression of the free oscillations are investigated in detail.

The total disturbance is generally regarded as an oscillation of forced frequency, the amplitude of which may be slowly variable.

If this amplitude is constant, the forced oscillation only is present. If in addition to the constant part of the amplitude, a slow and periodic variation of the amplitude is found, this would indicate simultaneous presence of free and forced oscillations. These oscillations react on one another due to the non-linear term containing γ in the equation, as opposed to cases of linear form in which this interaction is absent.

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